Quantum critical spin liquids and conformal field theory in $2+1$ dimensions

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Abstract

We describe new conformal field theories based on symplectic fermions that can be extrapolated between 2 and 4 dimensions. The critical exponents depend continuously on the number of components N of the fermions and the dimension D. In the context of anti-ferromagnetism, the $N=2$ theory is proposed to describe a deconfined quantum critical spin liquid corresponding to a transition between a Néel ordered phase and a VBS-like phase.

I. INTRODUCTION AND SUMMARY OF RESULTS

This paper concerns some new renormalization group (RG) fixed points in $D = d + 1 = 3$ dimensional quantum field theory based on symplectic fermions. In $1d$ there exists a vast assortment of critical points described by $2D$ conformal field theory[1]. However in $2d$ the known critical theories are up to now comparatively rare. Important examples are the Wilson-Fisher fixed points which are known to describe phase transitions in classical statistical mechanics in 3 spacial dimensions as a function of temperature[2, 3]. Further progress in some important $2d$ condensed matter systems, in particular superconductivity in the cuprates, has been hindered by the lack of understood critical points in $3D$.

This lengthy introduction will serve to summarize our main results. Since the motivation for our model initially came from quantum anti-ferromagnets in $2d$, we begin by reviewing the aspects of this problem that provide some perspective on our work. For a detailed account of this subject with additional references to the original works see Fradkin's book[4].

The Heisenberg hamiltonian for a collection of spins $\vec{S}_i$ on a d dimensional lattice is

$$H = J \sum_{<i,j>} \vec{S}_i \cdot \vec{S}_j \tag{1}$$

For the anti-ferromagetic case ($J > 0$), a continuum limit built over a staggered configuration close to the Néel state leads to the euclidean action

$$S = \frac{1}{2g} \int d^D x \; (\partial_\mu \vec{n} \cdot \partial_\mu \vec{n}) \quad + S_\theta \tag{2}$$

where $\vec{n}^{\,2} = 1$ and $\partial_\mu \partial_\mu = \sum_{\mu=1}^{D} \partial^2_{x_\mu}$. The latter constraint makes it an $O(3)$ non-linear sigma-model. In any dimension, the topological term S_θ arises directly in the map to the continuum when one formulates the functional integral for the spins $\vec{S}_i$ using coherent states[5, 6], and is related to the area swept out by the vector $\vec{n}$ on the 2-sphere. In $1d$ it is given explicitly by

$$S_\theta = \frac{\theta}{2\pi} \int d^2 x \; \epsilon_{\mu\nu} \vec{n} \cdot (\partial_\mu \vec{n} \times \partial_\nu \vec{n}) \tag{3}$$

with $\theta = 2\pi s$ where s is the spin of $\vec{S}$: $\quad \vec{S}_i^2 = s(s+1)$. When s is an integer, S_θ has no effect in the functional integral: the model has no non-trivial infra-red (IR) fixed point and the model is gapped. For half-integer spin chains, $S_{\theta=\pi}$ modifies the IR behavior and the model has an infra-red fixed point which necessarily has massless degrees of freedom. This is the well-known Haldane conjecture[7].

The above understanding of $1d$ spin chains relies to a considerable extent on the Bethe-ansatz solution[8], which shows massless excitations. What is more subtle, and was only realized much later, is that the low-lying excitations are actually spin 1/2 particles referred to as spinons[9]. These spinons are responsible for destroying the long-range Néel order. The $\vec{n}$ field is composed of two spinons, so this is a $1d$ precedent to the $2d$ deconfined quantum critical points discussed by Senthil et. al.[10]. In the latter work, numerous arguments were given that in $2d$ there should exist novel critical points that describe for example the transition between Néel order and a valence-bond solid (VBS) phase, and this idea strongly motivated our work initially.

In $2d$ the term S_θ does not appear to have a significant role. One way to anticipate this is that unlike in $1d$ where $\vec{n}$ and S_θ are classically dimensionless (in fact S_θ is exactly marginal), in $2d$ $\vec{n}$ has classical dimension 1/2 so that S_θ is already RG irrelevant before any quantum anomalous corrections.

Remarkably in $2d$ a non-trivial IR fixed point appears in the Heisenberg anti-ferromagnet that doesn't rely on the existence of S_θ and can be understood in the following way[11, 12]. The non-linear constraint $\vec{n}^{\,2} = 1$ renders the non-linear sigma model perturbatively non-renormalizable in $2d$. (Unlike in $1d$, see [13].) If a fixed point is understandable by Wilsonian RG, this non-renormalizability is potentially a serious problem. However the infra-red behavior is captured by the following scalar field theory:

$$S_{WF} = \int d^3x \left(\frac{1}{2} \partial_\mu \vec{n} \cdot \partial_\mu \vec{n} + \widetilde{\lambda}\, (\vec{n} \cdot \vec{n}\,)^2 \right) \tag{4}$$

where now $\vec{n}$ is not constrained to be a unit vector and hence is a linear sigma model but with interactions. The above model can be studied in the epsilon expansion around $D = 4$ and the IR fixed point is seen perturbatively. (See for instance [14, 15].) This is the universality class of the Wilson-Fisher (WF) fixed point, even though it is a quantum critical point[16] at zero temperature. We will refer to this model as the $O(M)$ linear sigma model and the WF fixed point conformal field theory as $O_M^{(D)}$ in $D < 4$ dimensions.

A large part of the literature devoted to the on-going search for other ground states of quantum spins represents the $\vec{n}$ field in terms of spinon fields z:

$$\vec{n} = z^\dagger \vec{\sigma} z \tag{5}$$

where $\vec{\sigma}$ are the Pauli matrices and $z = (z_1, z_2) = \{z_i\}$ is a two component complex bosonic spinor. The constraint $\vec{n}^{\,2} = 1$ then follows from the constraint $z^\dagger z = 1$. Coupling z to a

$U(1)$ gauge field A_μ with the covariant derivative $D_\mu = \partial_\mu - iA_\mu$, then by eliminating the non-dynamical gauge field using it's equations of motion, one can show that the following actions are equivalent:

$$\int d^D x \ \tfrac{1}{2}\partial_\mu \vec{n} \cdot \partial_\mu \vec{n} = \int d^D x \ |D_\mu z|^2 \tag{6}$$

(One needs $\vec{\sigma}_{ij} \cdot \vec{\sigma}_{kl} = 2\delta_{il}\delta_{jk} - \delta_{ij}\delta_{kl}$.)

Since the spinon fields have classical dimension 1/2, in terms of the spinon fields, the $(\vec{n} \cdot \vec{n}\,)^2$ term in the action (4) is a dimension 4 operator which is irrelevant in $2d$. In analogy with the WF fixed point, in order to deal with the non-renormalizability it is natural to relax the constraint $\vec{n}^{\,2} = |z^\dagger z| = 1$ and to consider $(z^\dagger z)^2$ terms of dimension 2 which are RG relevant. However the fixed point is then still in the universality class of the WF fixed point. In an effort to perturb the WF fixed point, the authors in [10] make the gauge field dynamical by adding $(F_{\mu\nu})^2$ and thus consider a QED-like theory (in $3D$). It was proposed that the model has a fixed point in the universality class of a hedgehog suppressed $O(3)$ sigma model, however because of the expected non-perturbative nature of the fixed point, it hasn't been possible to compute any of it's critical exponents. It has also proven difficult to see such a second-order transition in simulations of the model[17].

With the above background we now present the central idea of this paper. The WF fixed point in this anti-ferromagnetic context is a quantum critical confined phase since it is described in terms of the $\vec{n}$ degrees of freedom. A deconfined quantum critical point is defined then as one describable with the spinon z degrees of freedom. We will postulate that for a deconfined critical point the spinon field z should actually be a fermion field, henceforth denoted χ, and described by the action

$$S_\chi = \int d^D x \ \left(\partial_\mu \chi^\dagger \partial_\mu \chi + \widehat{\lambda} \ |\chi^\dagger \chi|^2 \right) \tag{7}$$

where χ is a two-component complex field, $\chi^\dagger \chi = \sum_{i=1,2} \chi_i^\dagger \chi_i$. The non-linear constraint $\chi^\dagger \chi = 1$ is obviously relaxed. Note that our model contains no gauge field. In terms of real fields each component can be written as $\chi = \eta_1 + i\eta_2$, $\chi^\dagger = \eta_1 - i\eta_2$ and the free action is

$$S_{\text{symplectic}} = i \int d^D x \ \epsilon_{ij} \, \partial_\mu \eta_i \partial_\mu \eta_j \tag{8}$$

where $\epsilon_{12} = -\epsilon_{21} = 1$. The $2N$ real component version has the symmetry $\eta \to U\eta$ where U is a $2N \times 2N$ dimensional matrix satisfying $U^T \epsilon_N U = \epsilon_N$ where $\epsilon_N = \epsilon \otimes 1_N$. This implies the theory has $Sp(2N)$ symmetry, hence is sometimes referred to as a symplectic fermion.

This kind of theory has a number of important applications in $1d$, for instance to dense polymers[18], and to disordered Dirac fermions in $2d$[19]. It is known to be a non-unitary theory and this potential difficulty will be addressed below.

Arguments suggesting that symplectic fermions are natural in this context are the following. First of all, in $1d$ the spinon is neither a boson nor a fermion but a semion, i.e. half fermion as far as its statistics is concerned, so there is a precedent for this kind of modified statistics. We remind the reader that there is no spin-statistics theorem for $2d$ relativistic theories since spin is not necessarily quantized. Note also that the identity (6) remains true if z is a fermion. Secondly, suppose the theory is asymptotically free in the ultra-violet. Then in this free, conformally invariant limit, one would hope that the description in terms of $\vec{n}$ or z are somehow equivalent, or at least have the same numbers of degrees of freedom. One way to count these degrees of freedom is to study the theory at finite temperature $T = 1/\beta$ and consider the free energy. For a single species of free massless particle the free energy density is

$$\mathcal{F} = \pm\frac{1}{\beta}\int \frac{d^d\vec{k}}{(2\pi)^d}\; \log\left(1 \mp e^{-\beta\omega_{\vec{k}}}\right) \tag{9}$$

where $\omega_{\vec{k}} = |\vec{k}|$ and the upper/lower sign corresponds to bosons/fermions. In $2d$

$$\mathcal{F} = -c_3\frac{\zeta(3)}{2\pi}T^3 \tag{10}$$

where $c_3 = 1$ for a boson and $3/4$ for a fermion. (In $1d$ the analog of the above is $\mathcal{F} = -c\pi T^2/6$, where c is the Virasoro central charge[20, 21].) Therefore one sees that the 3 bosonic degrees of freedom of an $\vec{n}$ field has the same c_3 as an $N = 2$ component χ field. This simple observation is what first pointed us in the direction of symplectic fermions. It suggests a kind of bosonization where 3 bosons are equivalent to 4 fermions. In $1d$, one boson is equivalent to 2 fermions, and one can explicitly construct the fermion fields in terms of bosons, but we won't need to attempt the analog here.

Lastly, and most importantly, the symplectic fermion theory has an infra-red stable fixed point that is not in the WF universality class, and this is the main subject of this paper. The exponents can be computed in the very low order epsilon expansion around $D = 4$ and they are in excellent agreement with the exponents found numerically for the hedgehog-free model studied by Motrunich and Vishwanath[22]. The agreement is better than we anticipated. We find

$$\eta = 3/4, \quad \nu = 4/5, \quad \beta = 7/10 \qquad (N = 2, D = 3) \tag{11}$$

compared to $\nu = .8 \pm 0.1$, $\beta/\nu = .85 \pm 0.05$. The shift down to 3/4 from the classical value $\eta = 1$ is entirely due to the fermionic nature of the χ fields. Since we are discussing a zero temperature critical point, temperature exponents are probably not important for comparison with experiments, so we also define a magnetic exponent δ so that $\langle \vec{n} \rangle \sim B^{1/\delta}$ where B is an external magnetic field and find $\delta = 17/7$. It is the fermionic statistics of χ that is also ultimately responsible for the largeness of these exponents in comparison with the bosonic WF fixed point. We thus conjecture that our 2-component model describes the fixed point in [22], which is believed to be a deconfined critical point of the kind discussed by Senthil et. al.[10]. It is important to point out that there are no "emergent photons" in our model.

Let us return now to the issue of topological terms. So far we have ignored the gauge field A_μ in the action (6). Wilczek and Zee have shown how to include a topological term[23] that is intrinsic to $2d$, which is a Chern-Simons or Hopf term:

$$S_{\rm CS} = \frac{\vartheta}{8\pi^2} \int d^3x \; \epsilon_{\mu\nu\lambda} A^\mu F^{\nu\lambda} \tag{12}$$

It is well-known that ϑ modifies the statistics of the z, and this provides an obvious mechanism for obtaining the fermionic statistics of the χ field when $\vartheta = \pi$. It should be emphasized that this Chern-Simons term has nothing to do with the S_θ term discussed above. Whereas the latter arises directly in the map to the continuum, the coefficient of the Chern-Simons term is more subtle. The consensus is that for a square lattice $\vartheta = 0$[6, 24, 25, 26, 27].

The idea that the fermionic χ model should correspond to the action (6) with the addition of a Chern-Simons term with $\vartheta = \pi$ helps to resolve the problem that the symplectic fermion theory is non-unitary, since the Chern-Simons description is thought to be unitary. In this description the χ particles are to be viewed as having π flux attached microscopically. We return to the issue of the non-unitarity in section IV and give a different possible resolution of it based on a simple projection of the Hilbert space onto pairs of particles.

In this paper we do not address what kind of microscopic theory can give rise to $\vartheta \neq 0$ and whether it is related for instance to the spin s of the $\vec{S}$, as in $\theta = 2\pi s$. But we can nevertheless state a $2d$ version of the Haldane conjecture: When $\theta = 0$ (mod 2π), the quantum critical point is confined and in the universality class of the WF fixed point. On the other hand when $\theta = \pi$ (mod 2π), the quantum critical point is in the different universality class of the fixed point in the $N = 2$ component fermionic theory described in this paper.

The bulk of this paper analyzes the RG for the symplectic model to support the above statements. In section II we compute the beta function in $D=4$ from the effective potential, and display the fixed point in $D<4$. In section III we define the critical exponents and relate them to anomalous dimensions in the symplectic fermion theory. The relevant correlation functions are computed to lowest orders in position space and this determines the exponents as a function of N and D. (In terms of Feynman diagrams, which we manage to avoid, this involves one and two loop diagrams. For the exponent ν one actually only needs 1-loop results.)

In section IV we study our model in a hamiltonian framework in momentum space and show how the non-unitarity in manifested. We argue that a simple projection onto pairs of particles renders the Hilbert space unitary. We also discuss possible applications to superconductivity in the cuprates.

II. RG BETA FUNCTION

A. Functional RG

The 3D fixed point can studied systematically in the epsilon expansion around $D=4$[14, 15]. The Feynman diagram techniques developed for the WF fixed point, namely dimensional regularization, lagrangian counterterms, etc, is easily generalized to our fermionic theory. However since we will work to lowest orders only, we need not develop the epsilon expansion in much detail since the required quantities can be computed directly in $4D$. Working with position space correlation functions also helps to clarify the physical content.

For the beta function it turns out to be convenient to work with the Coleman-Weinberg effective potential. This avoids Feynman diagrams and can be especially useful if the potential has complicated anisotropy, though this will not be investigated here. It also helps to keep track of the all-important fermionic minus signs.

Let χ denote a vector of complex fermionic (Grassman) fields $\chi=(\chi_1,\ldots,\chi_N)=\{\chi_i\}$, and consider the euclidean action:

$$S_\chi=\int d^Dx\,\left(2\partial_\mu\chi^\dagger\partial_\mu\chi+U(\chi^\dagger,\chi)\right) \tag{13}$$

where U is the potential. For the purposes of Grassman functional integration, it is convenient to arrange $\chi,\chi^\dagger$ into a $2N$ dimensional vector $\Psi=(\chi,\chi^\dagger)$. Note that $\Psi^\dagger=\Psi^T\Sigma_1$

where in block form $\Sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, so that $\Psi^\dagger$ is not independent of Ψ as far as functional integration is concerned.

Consider the functional integral

$$Z = \int D\Psi \; e^{-S} \tag{14}$$

in a saddle point approximation. Expanding Ψ around Ψ_0 and performing the fermionic gaussian functional using the formula

$$\int D\Psi \; e^{-\Psi^\dagger A \Psi} = \sqrt{\mathrm{Det} A} \tag{15}$$

one obtains $Z \sim e^{-S_{\mathrm{eff}}}$ where the effective action is

$$S_{\mathrm{eff}} = S(\Psi_0) - \frac{1}{2} \mathrm{Tr} \log A \tag{16}$$

and the Trace is functional. The operator A is

$$A = \begin{pmatrix} -\partial^2 & 0 \\ 0 & \partial^2 \end{pmatrix} - \frac{1}{2} U'' \tag{17}$$

where U'' is a matrix of second derivatives:

$$U'' = \begin{pmatrix} \frac{\partial^2 U}{\partial \chi^\dagger \partial \chi} & \frac{\partial^2 U}{\partial \chi^\dagger \partial \chi^\dagger} \\ \frac{\partial^2 U}{\partial \chi \partial \chi} & \frac{\partial^2 U}{\partial \chi \partial \chi^\dagger} \end{pmatrix} \tag{18}$$

Assuming U'' is constant, the trace can be computed in momentum space. Dividing by the volume one defines the effective potential

$$V_{\mathrm{eff}} = U - \frac{1}{2} \mathrm{tr} \int \frac{d^D k}{(2\pi)^D} \; \log \left(\begin{pmatrix} k^2 & 0 \\ 0 & -k^2 \end{pmatrix} - \frac{1}{2} U'' \right) \tag{19}$$

Since we are interested only in the term that determines the beta function in $D = 4$, we expand the log to second order in U'' which involves $1/k^4$. Performing the integral over k with an ultra-violet cut off μ:

$$\int^\mu \frac{d^4 k}{(2\pi)^4} \frac{1}{k^4} = \frac{\log \mu}{8\pi^2} \tag{20}$$

one finds

$$V_{\mathrm{eff}} = U + \frac{\log \mu}{128\pi^2} \; \mathrm{tr}(\Sigma_3 U'' \Sigma_3 U'') \tag{21}$$

where in block form

$$\Sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{22}$$

The renormalization group of the potential then follows from requiring

$$V_{\text{eff}}(U,\mu) = V_{\text{eff}}(U(a), e^a\mu) \tag{23}$$

We adopt the usual convention in statistical physics and define the beta function as the flow toward the infra-red, i.e. to low energies: $dU/d\ell = -dU/da$ where e^ℓ is a length scale. Finally one gets the beta function:

$$\frac{dU}{d\ell} = \frac{1}{128\pi^2}\text{tr}(\Sigma_3 U''\Sigma_3 U'') \tag{24}$$

The flow toward the IR then corresponds to increasing ℓ.

The RG flow equation (24) is a functional RG since it determines the flow of the *entire* potential. It is then clear that not all potentials are renormalizable: only if $\text{tr}(\Sigma_3 U''\Sigma_3 U'')$ is proportional to U is the theory closed under RG flows. If additional terms are generated in the trace, they must be included in U until one obtains something renormalizable. By "renormalizable" we here mean that the flow just amounts to the flow of some couplings.

B. Beta function and fixed point

Let us take the following normalization of the coupling λ

$$U_\lambda = 16\pi^2\lambda\, U, \qquad U \equiv |\chi^\dagger\chi|^2 \tag{25}$$

Denoting $\partial_i = \partial/\partial\chi_i$, $\partial_i^\dagger = \partial/\partial\chi^\dagger$ one has

$$\begin{aligned} \partial_j\partial_i U &= -2\chi_i^\dagger\chi_j^\dagger \\ \partial_j^\dagger\partial_i U &= -2\delta_{ij}\chi^\dagger\chi + 2\chi_i^\dagger\chi_j \end{aligned} \tag{26}$$

Evaluating the trace in eq. (24) one finds that it is proportional to U since U was chosen to be isotropic. We also need that the classical dimension of χ is $(D-2)/2$ which implies the dimension of λ is $4-D$. Since the leading linear term always has a slope equal to the classical dimension of λ, the beta function is

$$\frac{d\lambda}{d\ell} = (4-D)\lambda + (N-4)\lambda^2 \tag{27}$$

The above beta function has a zero at

$$\lambda_* = \frac{4-D}{4-N} \tag{28}$$

Note that λ_* changes sign at $N = 4$. It is not necessarily a problem to have a fixed point at negative λ since the particles are fermionic: the energy is not unbounded from below because of the Fermi sea. Near λ_* one has that $d\lambda/d\ell \sim (D-4)(\lambda - \lambda_*)$ which implies the fixed point is IR stable regardless of the sign of λ_*, so long as $D < 4$. When $D > 4$ we have a short distance fixed point that is not asymptotically free.

III. SCALING THEORY AND CRITICAL EXPONENTS

There are two aspects of the scaling theory that differ from the usual WF fixed point for classical phase transitions. The first is that $\vec{n}$ is a composite field in terms of the χ's. The second is that due to the fermionic nature of χ, some of the exponents are in fact *negative*.

A. Definition of the exponents for the $\vec{n}$ field

Though the spinons χ are deconfined, it is still physically meaningful to define exponents in terms of the original order parameter $\vec{n}$, which is represented by

$$\vec{n} = \chi^\dagger \vec{\sigma} \chi \tag{29}$$

When $N = 2$, $\vec{\sigma}$ are the Pauli matrices; for general N they should be taken as representations of the Clifford algebra appropriate to spinor representations of $O(N)$, however we will not need these details in this paper. We then define the exponent η as the one characterizing the spin-spin correlation function:

$$\langle \vec{n}(\mathbf{x}) \cdot \vec{n}(0) \rangle \sim \frac{1}{|\mathbf{x}|^{D-2+\eta}} \tag{30}$$

For the other exponents we need a measure of the departure from the critical point; these are the parameters that are tuned to the critical point in simulations and experiments:

$$S_\chi \to S_\chi + \int d^D x \, (m^2 \chi^\dagger \chi + \vec{B} \cdot \vec{n}) \tag{31}$$

Above, m is a mass and $\vec{B}$ the magnetic field. For classical temperature phase transitions, typically $m^2 \propto (T - T_c)$. The correlation length exponent ν, and magnetization exponents β, δ are then defined by

$$\xi \sim m^{-\nu}, \qquad \langle \vec{n} \rangle \sim m^\beta \sim B^{1/\delta} \tag{32}$$

Above $\langle \vec{n} \rangle$ is the one-point function of the field $\vec{n}(\mathbf{x})$ and is independent of $\mathbf{x}$ by the assumed translation invariance.

To streamline the discussion, let $[\![X]\!]$ denote the scaling dimension of X in energy units, including the non-anomalous classical contribution which depends on D. An action S necessarily has $[\![S]\!]$ =0. Using $[\![d^D\mathbf{x}]\!] = -D$, the classical dimensions of couplings and fields are determined from $[\![S]\!] = 0$. The above exponents are then functions of $[\![\vec{n}]\!]$ and $[\![m]\!]$. Since $[\![\xi]\!] = -1$, one has $\nu = -[\![\xi]\!]/[\![m]\!] = 1/[\![m]\!]$. This, together with eq. (30), implies

$$\eta = 2[\![\vec{n}]\!] + 2 - D, \qquad \nu = 1/[\![m]\!] \tag{33}$$

One also has

$$\beta = [\![\vec{n}]\!]/[\![m]\!] = \frac{\nu}{2}(D - 2 + \eta) \tag{34}$$

The magnetic exponent is treated similarly. Treating $\vec{B}$ as a coupling, then $[\![\vec{B}]\!]+[\![\vec{n}]\!] = D$, which implies

$$\delta = \frac{D - [\![\vec{n}]\!]}{[\![\vec{n}]\!]} = \frac{D + 2 - \eta}{D + \eta - 2} \tag{35}$$

B. Relation to χ field exponents

The fundamental exponents of the symplectic fermion theory are the anomalous dimensions γ_χ, γ_m of χ and m, defined as:

$$[\![\chi]\!] \equiv (D-2)/2 + \gamma_\chi, \qquad [\![m]\!] \equiv 1 - \gamma_m \tag{36}$$

The γ_χ exponent determines the two point function of the χ fields:

$$\langle \chi^\dagger(\mathbf{x})\chi(0)\rangle \sim \frac{1}{|\mathbf{x}|^{D-2+2\gamma_\chi}}, \tag{37}$$

whereas ν is completely determined by γ_m:

$$\frac{1}{\nu} = 1 - \gamma_m \tag{38}$$

The scaling dimension $[\![\vec{n}]\!]$ is not a simple function of γ_χ to all orders since $\vec{n}$ must be treated as a composite operator. However, since $\vec{n}$ is quadratic in χ, let us assume that to lowest order $[\![\vec{n}]\!] = 2[\![\chi]\!]$. With this assumption one has

$$\eta = D - 2 + 4\gamma_\chi \tag{39}$$

C. Computation of correlation functions

To compute γ_χ, γ_m we need to consider the following correlation functions in $4D$. Introduce an ultra-violet cut-off μ as follows $\int d^4\mathbf{x} \to \int_{1/\mu} d^4\mathbf{x}$. Suppose the one-point function of $\chi^\dagger\chi$ satisfies

$$\langle \chi^\dagger \chi \rangle \sim m^2(1 - 2\alpha \log \mu) \approx (m\mu^{-\alpha})^2 \tag{40}$$

Then the ultra-violet divergence is canceled by letting $m \to m(\mu) = m\mu^\alpha$. This implies

$$\gamma_m = \alpha \tag{41}$$

This is equivalent to the statement $[\![\chi^\dagger\chi]\!] = D - 2 + 2\gamma_m$.

The exponent γ_χ is determined from the two-point function. Suppose

$$\langle \chi_i^\dagger(\mathbf{x})\chi_j(0)\rangle \sim \frac{\delta_{ij}}{|\mathbf{x}|^2}(1 - 2\alpha' \log|\mathbf{x}|) \approx \frac{\delta_{ij}}{|\mathbf{x}|^{2+2\alpha'}} \tag{42}$$

Then

$$\gamma_\chi = \alpha' \tag{43}$$

We now describe the lowest order contributions to the needed correlation functions. One needs the free 2-point functions:

$$\langle \chi_i^\dagger(\mathbf{x})\chi_j(0)\rangle = -\langle \chi_i(\mathbf{x})\chi_j^\dagger(0)\rangle = -\frac{\delta_{ij}}{8\pi^2|\mathbf{x}|^2} \qquad (\text{when } \lambda = 0) \tag{44}$$

Expanding the functional integral perturbatively in λ, to first order one has:

$$\langle \chi^\dagger\chi\rangle_\lambda = \langle \chi^\dagger\chi\rangle_0 - 16\pi^2\lambda \int d^4\mathbf{y}\ \langle \chi^\dagger\chi(0) U(\mathbf{y}))\rangle \tag{45}$$

Using Wick's theorem, the integrand goes as $1/|\mathbf{y}|^4$ and the y-integration gives $\log\mu$:

$$\langle \chi^\dagger\chi\rangle_\lambda = \Big(1 - \lambda(1-N)\log\mu\Big)\langle \chi^\dagger\chi\rangle_0 \tag{46}$$

From dimensional analysis $\langle \chi^\dagger\chi\rangle_{\lambda=0} \propto m^2$. Thus

$$\gamma_m = \frac{\lambda(1-N)}{2} \tag{47}$$

The negative sign in $1 - N$ in γ_m is entirely statistical in origin.

To compute γ_χ we need to go to second order:

$$\begin{aligned}\langle \chi_i^\dagger(\mathbf{x})\chi_j(0)\rangle_{\lambda^2} &= \tfrac{1}{2}(16\pi^2\lambda)^2 \int d^4\mathbf{y} \int d^4\mathbf{z}\ \langle \chi_i^\dagger(\mathbf{x})\chi_j(0)\, U(\mathbf{y})\, U(\mathbf{z})\rangle \\ &= \delta_{ij}\frac{(1-N)\lambda^2}{32\pi^6}\ I(|\mathbf{x}|)\end{aligned} \tag{48}$$

where $I(|\mathbf{x}|)$ is the integral:

$$I(|\mathbf{x}|) = \int d^4\mathbf{y} \int d^4\mathbf{z} \; \frac{1}{|\mathbf{y}|^2|\mathbf{x}-\mathbf{z}|^2|\mathbf{y}-\mathbf{z}|^6} \tag{49}$$

It is evaluated in Appendix A where it is shown:

$$I(|\mathbf{x}|) = \frac{2\pi^4 \log|\mathbf{x}|}{|\mathbf{x}|^2} \tag{50}$$

This in turn implies:

$$\gamma_\chi = \frac{(1-N)\lambda^2}{4} \tag{51}$$

D. Values of exponents for arbitrary N, D.

At the fixed point, one substitutes $\lambda = \lambda_*$ into the above expressions. The χ field exponents are:

$$\gamma_m = \frac{(4-D)(1-N)}{2(4-N)}, \qquad \gamma_\chi = \frac{(4-D)^2(1-N)}{4(4-N)^2} \tag{52}$$

which in turn imply the following $\vec{n}$ exponents:

$$\begin{aligned} \nu &= \frac{2(4-N)}{(2-D)N+D+4} \\ \beta &= \frac{2(D-2)(N^2-4N+12)+D^2(1-N)}{(4-N)(D(1-N)+2(N+2))} \end{aligned} \tag{53}$$

The η, δ exponents are given in terms of β, ν in eqs. (34,35).

For $N = 2$, the above expressions give the results quoted in the introduction: $\nu = 4/5, \beta = 7/10, \eta = 3/4$ and $\delta = 17/7$.

IV. HAMILTONIAN DESCRIPTION AND UNITARY PROJECTION

In this section we give a hamiltonian description of our model in momentum space and address the non-unitarity issue in a different way than in the Introduction. We also suggest how our work may provide some clues toward the understanding of superconductivity in the cuprates, which is believed to be a $2+1$ dimensional problem[31]. To do this, one must turn to the language of the Hubbard model. In the anti-ferromagnetic phase of the Hubbard model, the spin field $\vec{n} = c^\dagger \vec{\sigma} c$, where c are the physical electrons. Therefore in applying our model to the Hubbard model, the symplectic fermion χ is a descendant of

the electron, so it seems it can carry electric charge. Consider the zero temperature phase diagram of the cuprates as a function of the density of holes. At low density there is an anti-ferromagnetic phase. Suppose that the first quantum critical point is a transition from a Néel ordered to a VBS-like phase and is well described by our symplectic fermion model at $N = 2$. Compelling evidence for a VBS like phase has recently been seen by Davis' group[32]; and it in fact resembles more a "VBS spin glass". The superconducting phase actually originates from this VBS-like phase. Beyond the first deconfined quantum critical point at higher density, it is then possible that the 2-component χ fields capture the correct degrees of freedom for the description of this VBS-like phase. It is important to point out that we are imagining that the spinons are still deconfined in the VBS-like phase, in contrast to ideas in [10]. These fermionic spinon quasi-particles acquire a gap away from the critical point, which is described by the mass term in eq. (31). Note that away from the quantum critical point, the particles already have a gap m because of the relativistic nature of the symplectic fermion.

Particles with a relativistic kinetic energy are actually not entirely new in $2d$ physics: graphene appears to have massless particles described by the Dirac equation[29, 30], rather than the symplectic fermion theory we considered. In graphene the origin of the massless Dirac equation is simply band theory on a hexagonal lattice, so the origin of massless relativistic particles is entirely different (and much simpler) than the origin of our symplectic fermions.

The hamiltonian of the symplectic fermion is

$$H = \int d^2\mathbf{x} \left(2\partial_t \chi^\dagger \partial_t \chi + 2\vec{\nabla}\chi^\dagger \cdot \vec{\nabla}\chi + m^2 \chi^\dagger \chi + \widetilde{\lambda}\ (\chi^\dagger \chi)^2 \right) \tag{54}$$

Expand the field in terms of creation/annihilation operators as follows

$$\chi(\mathbf{x}) = \int \frac{d^2\mathbf{k}}{4\pi\sqrt{\omega_{\mathbf{k}}}} \left(a_{\mathbf{k}}\, e^{-i\mathbf{k}\cdot\mathbf{x}} + b_{\mathbf{k}}\, e^{i\mathbf{k}\cdot\mathbf{x}} \right) \tag{55}$$

and similarly for $\chi^\dagger$, where $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$. Canonical quantization of the χ-fields leads to

$$\{b^\dagger_{\mathbf{k}}, b_{\mathbf{k}'}\} = -\{a^\dagger_{\mathbf{k}}, a_{\mathbf{k}'}\} = \delta_{\mathbf{k},\mathbf{k}'} \tag{56}$$

The free hamiltonian is then

$$H_0 = \int d^2\mathbf{k}\ \omega_{\mathbf{k}}\ \left(a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + b^\dagger_{\mathbf{k}} b_{\mathbf{k}} \right) \tag{57}$$

Because of the minus sign in eq. (56) this is a two-band theory:

$$\begin{aligned} H_0 \; b^\dagger_{\mathbf{k}} |0\rangle &= \omega_{\mathbf{k}} \; b^\dagger_{\mathbf{k}} |0\rangle \\ H_0 \; a^\dagger_{\mathbf{k}} |0\rangle &= -\omega_{\mathbf{k}} \; a^\dagger_{\mathbf{k}} |0\rangle \end{aligned} \tag{58}$$

A two-band structure of this kind has been observed experimentally [32] and in the Hubbard model[33].

The unconventional minus sign in the anti-commutator of the $a, a^\dagger$ is a manifestation of the non-unitarity. In particular it implies that the state $|\mathbf{k}\rangle_a = a^\dagger_{\mathbf{k}}|0\rangle$ has negative norm: ${}_a\langle \mathbf{k}|\mathbf{k}'\rangle_a = -\delta_{\mathbf{k}\mathbf{k}'}$. On the other hand, the two-particle state $|\mathbf{k}_1, \mathbf{k}_2\rangle_a$ has positive norm. We thus propose to resolve the non-unitarity problem by simply projecting the Hilbert space onto even numbers of a-particles. It is clear that if they arise from deconfinement of the $\vec{n}$ field, they will be created in pairs.

Let us turn to the interactions. In the VBS-like phase the χ-particles are charged fermions and it's possible that additional phonon interactions, or even the χ^4 interactions that led to the critical theory, could lead to a pairing interaction that causes them to condense into Cooper pairs just as in the usual BCS theory. Recent numerical work on the Hubbard model suggests that the Hubbard interactions themselves can provide a pairing mechanism[33]. The $(\chi^\dagger \chi)^2$ interaction is very short ranged since it corresponds to a δ-function potential in position space. Because of the relativistic nature of the fields, the interaction gives rise to a variety of pairing interactions. Let us examine pairing interactions within each band that resemble BCS pairing. If all momenta have roughly the same magnitude $|\mathbf{k}|$, then the interaction gives the term (up to factors of π):

$$H_{\rm int} = -\widetilde{\lambda} \sum_{\mathbf{k};\; i,j=\uparrow,\downarrow} \left(a^\dagger_{\mathbf{k},i} a^\dagger_{-\mathbf{k},j} a_{-\mathbf{k},i} a_{\mathbf{k},j} + (a \to b) \right) + ... \tag{59}$$

The overall minus sign of the interaction is due to the fermionic statistics. To compare with the BCS theory, the interaction contains terms such as

$$H_{\rm int} = -\widetilde{\lambda} \sum_{\mathbf{k}} \left(a^\dagger_{\mathbf{k}\uparrow} a^\dagger_{-\mathbf{k}\downarrow} a_{-\mathbf{k}\uparrow} a_{\mathbf{k}\downarrow} \right) + ... \tag{60}$$

Because of the overall minus sign this is an attractive pairing interaction as in BCS. One difference is that some pairing interactions have the spins flipped in comparison with BCS.

This discussion has been expanded and a few more results derived in [34].

V. CONCLUSIONS

We have described new $3D$ fixed points based on symplectic fermions and have applied the $N = 2$ case to deconfined quantum criticality. In this interpretation, the symplectic fermions are the deconfined spinons. The main evidence for our model is the agreement with the exponents found in [22] for the hedgehog-free $O(3)$ sigma model. The fermionic nature of the χ-field is what is responsible for the negative contributions to exponents like η that bring it below the classical value $\eta = 1$.

Our model is non-unitary, however we have addressed this in two ways, one based on a unitary Chern-Simons description, the other based on a projection of the Hilbert space onto even numbers of particles.

After this work was completed, new simulations by Sandvik report evidence for a deconfined quantum critical point in a Heisenberg model with four-spin interactions. There it was found that $\nu = 0.78 \pm 0.03$ consistent with [22] and with our prediction of $\nu = 4/5$. The η exponent on the other hand, $\eta = 0.26 \pm 0.03$ is quite different from both our result and the one in [22]. This could simply mean that the critical point in the four-spin interaction model is in a different universality class. It could also mean there are significant corrections to our exponents at higher order or due to the compositeness of $\vec{n}$; recall the fixed point at lowest order occurs at the relatively large value $\lambda_* = 1/2$. Higher order computations are currently in progress and we hope to report our results in the near future.

Our exponents are well defined for any $N < 4$, including negative N. Based on the comparison of exponents, our model for N a negative integer was proposed to describe $O(M)$ models such as the Ising model in [36] with the identification $M = -N$.

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VII. APPENDIX A

In this appendix we do the integral $I(|\mathbf{x}|)$ in eq. (49). Shifting $\mathbf{z} = \mathbf{z}' + \mathbf{y}$, and using the identity

$$\frac{1}{AB} = \int_0^1 dt \, \frac{1}{(tA + (1-t)B)^2} \tag{61}$$

one can do the integral over $\mathbf{y}$:

$$\int_0^L d^4\mathbf{y} \, \frac{1}{|\mathbf{y}|^2 |\mathbf{y} - \mathbf{w}|^2} = \pi^2 \left(\log(|\mathbf{w}|^2/L^2) - 3 \right) \tag{62}$$

where $\mathbf{w} = \mathbf{x} - \mathbf{z}'$ and L is an IR cut-off. Introducing an ultra-violet cut-off μ, $\int d^4\mathbf{x} \to \int_{\mu^{-1}}^L d^4\mathbf{x}$ the integral over $\mathbf{z}'$ can be performed giving

$$I(|\mathbf{x}|) = \frac{\pi^4}{|\mathbf{x}|^2} \left(2 \log(|\mathbf{x}|/L) + 1 \right) - 6\pi^4 \mu^2 \tag{63}$$

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